Tutorial 3 (Feb 18, 20)

Leon Li

ylli@math.cuhk.edu.hk

Q1) Find the average value of the function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$

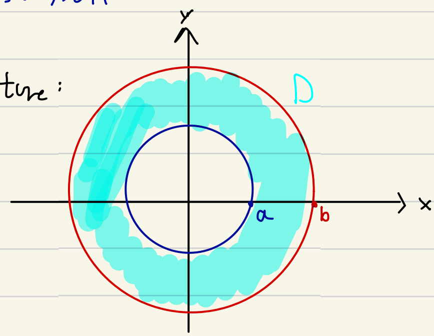
over the annular region $D = \{(x,y) \in \mathbb{R}^2 \mid a^2 \leq x^2+y^2 \leq b^2\}$, where $0 < a < b$.

Sol) Idea: Compute two relevant double integrals using polar coordinates.

By definition, average of f over $D = \frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dA$

Step 1: Sketch D.

Picture:



Step 2: Describe D in terms of polar coordinates

$$D = \{(r,\theta) \in (0,+\infty) \times [-\pi,\pi) \mid a \leq r \leq b\}$$

Step 3: Compute Area(D).

Method 1: By double integral: $\text{Area}(D) = \iint_D 1\,dA$

$$= \int_{-\pi}^{\pi} \int_a^b r\,dr\,d\theta = \int_{-\pi}^{\pi} \left[\frac{r^2}{2}\right]_a^b d\theta = \int_{-\pi}^{\pi} \left(\frac{b^2-a^2}{2}\right) d\theta = \left[\frac{b^2-a^2}{2}\theta\right]_{-\pi}^{\pi} = \pi(b^2-a^2)$$

Method 2: By elementary geometry: $\text{Area}(D) = \text{Area}(\ ) - \text{Area}(\ )$

$$= \pi b^2 - \pi a^2 = \pi(b^2-a^2)$$

Step 4: Compute $\iint_D f(x,y)\,dA$ using polar coordinates.

then $f(r,\theta) = \frac{1}{\sqrt{r^2}} = \frac{1}{r}$

$$\therefore \iint_D f(x,y)\,dA = \int_{-\pi}^{\pi}\int_a^b \frac{1}{r}\cdot r\,dr\,d\theta$$

$$= \left(\int_{-\pi}^{\pi} d\theta\right)\cdot\left(\int_a^b dr\right) = 2\pi\cdot(b-a)$$

Step 5: Compute the average of f over D.

$$\text{Average} = \frac{1}{\text{Area}(D)}\iint_D f(x,y)\,dA = \frac{1}{\pi(b^2-a^2)}\cdot 2\pi(b-a) = \frac{2}{b+a}$$

Q2) Evaluate $\underbrace{\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx}_{\mathrm{I}} + \underbrace{\int_{1}^{\sqrt{2}}\int_{0}^{x} xy\,dy\,dx}_{\mathrm{II}} + \underbrace{\int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xy\,dy\,dx}_{\mathrm{III}}$.

Sol) Method 1: Directly compute each integral.

$$\mathrm{I}+\mathrm{II}+\mathrm{III} = \int_{\frac{1}{\sqrt{2}}}^{1}\left[\frac{xy^2}{2}\right]_{\sqrt{1-x^2}}^{x} dx + \int_{1}^{\sqrt{2}}\left[\frac{xy^2}{2}\right]_{0}^{x} dx + \int_{\sqrt{2}}^{2}\left[\frac{xy^2}{2}\right]_{0}^{\sqrt{4-x^2}} dx = \cdots = \frac{15}{16}$$

Method 2: Combining three integrals into one by merging the domains.

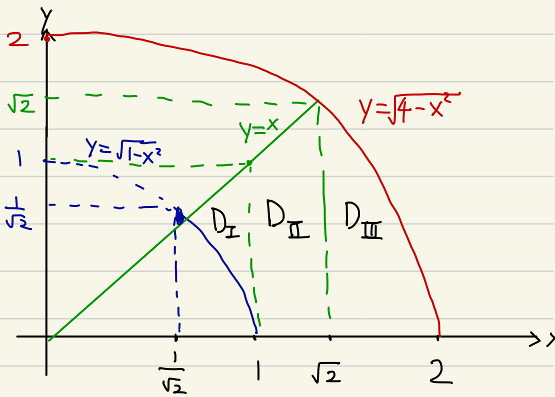
Step 1: Sketch the domains of integration.

$D_{\mathrm{I}} = \{(x,y)\in\mathbb{R}^2 \mid \frac{1}{\sqrt{2}} \le x \le 1,\ \sqrt{1-x^2} \le y \le x\}$

$D_{\mathrm{II}} = \{(x,y)\in\mathbb{R}^2 \mid 1 \le x \le \sqrt{2},\ 0 \le y \le x\}$

$D_{\mathrm{III}} = \{(x,y)\in\mathbb{R}^2 \mid \sqrt{2} \le x \le 2,\ 0 \le y \le \sqrt{4-x^2}\}$

Picture:



Step 2: Merge the domains.

$D := D_{\mathrm{I}} \cup D_{\mathrm{II}} \cup D_{\mathrm{III}} = \{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{4};\ 1 \le r \le 2\}$

polar coordinates

Step 3: Compute the integral over D.

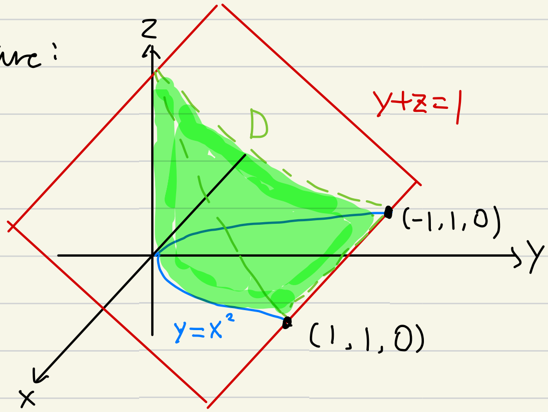
$$\mathrm{I}+\mathrm{II}+\mathrm{III} = \iint_D xy\,dA = \int_0^{\frac{\pi}{4}}\int_1^2 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta$$

$$= \left(\int_0^{\frac{\pi}{4}} \cos\theta\sin\theta\,d\theta\right)\cdot\left(\int_1^2 r^3\,dr\right) = \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{4}} \cdot \left[\frac{r^4}{4}\right]_1^2 = \frac{1}{4}\cdot\frac{15}{4} = \frac{15}{16}$$

Q3) Find the volume of the solid enclosed by the cylinder $y=x^2$ and the planes $z=0$, $y+z=1$.

Sol) Idea: Compute the volume by a triple integral

Step 1: Sketch the solid D. Picture:



Step 2: Express D in coordinates.

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; -1 \le x \le 1,\ x^2 \le y \le 1,\ 0 \le z \le 1-y \right\}$$

Step 3: Compute the volume of D by triple integral.

$$\text{Volume} = \iiint_D 1 \cdot dV = \int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} [z]_0^{1-y}\,dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} (1-y)\,dy\,dx = \int_{-1}^{1} \left[y - \frac{y^2}{2}\right]_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \left(\left(1-\frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right)\right) dx = \int_{-1}^{1} \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx$$

$$= 2\int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx = 2\left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) = \frac{8}{15}$$

Remark: Alternatively, the volume can be computed by a double integral

$$\iint_{D'} (1-y)\,dA, \text{ where } D' = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1,\ x^2 \le y \le 1\}.$$